Reconstructed CKM matrices

Reconstructed CKM matrices

K J Barnes, O J Senior and N D Virgo

Department of Physics, University of Southampton, Southampton SO17 4JU, UK

Abstract. We construct quark mixing matrices within a group theoretic framework which is easily applicable to any number of generations. Familiar cases are retrieved and related, and it is hoped that our viewpoint may have advantages both phenomenologically and for constructing underlying mass matrix schemes.

At a recent meeting in Meribel one of the authors (KJB) was struck by a particular presentation by Kayser [1, 2] and the reaction of many participants in the audience. The topic was that of describing the Cabibbo–Kobayashi–Maskawa (CKM) matrix [3–7] in terms of 4 phases of the unitarity triangles [8], and the connection with CP violation. Although the audience might well be considered 'expert', there was a marked resistance to consider seriously anything other than formulations (presumably many and varied) with which participants were already working, and the questions and comments revealed clear misunderstandings of other schemes and parametrizations. This brief note is an attempt to encourage a wider appreciation of the parametrizations of the CKM matrix and the connections between them in a very simple manner. It is directly applicable to larger numbers of generations of quarks should this turn out to be forced by physics in the future.

In the standard model with n generations the CKM matrix appears as an $n \times n$ unitary matrix, $(V_n)_\alpha^\beta$, mixing the n left-handed lower weak isospin quarks, D_β, to reflect the change of basis of the quarks from current eigenstates to mass eigenstates. The charged current then couples these to the adjoint of the n left-handed higher weak isospin mass eigenstates $\overline{U}^\alpha$. This notation is chosen to emphasize the group theoretic $U(n)$ nature of the structure, although no physical symmetry need be ascribed to this group. In particular, it should be noted that this group should not be confused either with the weak $SU(2) \times U(1)$ group, nor with the earlier hadronic organising $SU(3)$ group of Gell-Mann or its extensions to $SU(4)$, etc. Nevertheless, from the point of view of parametrizing the CKM matrix the Gell-Mann [9] notation will prove to be very convenient. Thus the D_α are assigned to the n dimensional fundamental representation, as indeed are the U_α so that $\overline{U}^\alpha$ are in the conjugate multiplet. Thus $(V_n)_\alpha^\beta$ may be expanded in any general unitary form (such as exponential) in terms of the adjoint multiplet represented by the $(n^2 - 1)$ Gell-Mann ($n \times n$) Hermitian $\underline{\lambda}$ matrices and the unit. This makes clear that V_n contains n^2 real parameters, but since the relative phases of the elements of D_α and $\overline{U}^\beta$ can be independently picked $(2n - 1)$ of the parameters can be removed (the overall phase is irrelevant) and $(n - 1)^2$ independent real parameters suffice. We shall see later in particular cases how these give rise in general to both real and complex elements of V, and are frequently interpreted as 'mixing angles' and 'complex phases'. These latter give rise to the possibility of CP violation if $n > 2$. In other

interpretations the parameters are viewed as 'phases' of unitarity triangles. The physics is, of course, independent of these interpretations although they may be useful for visualization of the phenomenology, or lead to intuitions as to the underlying mass mechanisms, and should not be undervalued.

We will now show how the λ matrix framework is directly relevant to implementing the phase freedoms mentioned above, and actually constructing useful parametrizations of V_n. It is convenient to recall that the Gell-Mann representation of the λ matrices can be built up inductively as n increases. Since the rank of $SU(n)$ is $(n - 1)$, there are $(n - 1)$ diagonal traceless matrices designated $\lambda_{k(k+2)}$, with $k = 1$ to $n-1$, where $\lambda_{k(k+1)}$ has entries $\sqrt{2/k(k-1)}$ down the first $(k - 1)$ diagonal places, $(-)\sqrt{2(k - 1)/k}$ in the next diagonal place, and zeros in all other places, so that the trace of its square is two.

The off diagonal matrices follow the pattern of the Pauli matrices both in terms of ordering sequence and entries. Thus λ_1 has a 1 in the first row and second column, and λ_2 has a $-i$ in the same place, all other entries being zero except for the complex conjugate entries to the forementioned in the transposed matrix position to ensure hermiticity. The traceless property is obvious, as is the continuation of the normalization that the trace of each square is two. Clearly the number of off diagonal matrices thus constructed to be listed as λ_i, with $(k - 1)^2 \leqslant i \leqslant k^2 - 2$, is $2(k - 1)$, since the entries 1 and $-i$ are placed sequentially in the rows of the kth column starting in the first row and going down to the row $(k - 1)$ as k increases through the range specified. It follows that the number of off diagonal matrices in all is $n(n - 1)$, and these together with the $(n - 1)$ diagonal matrices yield the full basis of $n^2 - 1$ traceless Hermitian matrices.

We are now in a position to make a preliminary discussion of what is meant by a parametrization of V_n, at this stage not considering the removal of phases and so dealing with n^2 real parameters. Obviously, since we have a basis of n^2 Hermitian matrices, comprising the λ_i and the unit, a familiar unitary parametrization is available in the form

$$V_n = \exp\left[\frac{-i}{2}(\theta_i \lambda^i + \chi 1)\right] \tag{1}$$

where the n^2 parameters, θ_i and χ, are all real. Of course, many other unitary constructions are possible, and these include products of several factors each of which are unitary (usually exponential). Two conditions must be observed. The parameters are essential in the technical sense, so that there must be n^2 of them. If the expansion of a given parametrization for small parameters coincides with the expansion of equation (1), then these may be viewed as equivalent. (We ignore parametrizations at large values of the angles.) But this is not the only possibility. It is also acceptable if the expansion involves only a subset of the matrices, but these yield the full set of matrices under repeated commutation. This will include important known cases of parametrization as will be demonstrated shortly in particular examples, but the idea may already be familiar to the reader through the Euler angle specification of rotations in three dimensions. There the three rotations are not about three independent orthogonal axes, but two are about a single axis with the third separating these two being about a second axis. The expansion for small angles only contains two independent infinitesimal generators (although three parameters are used) but these commute to produce the third infinitesimal generator as the A_1 algebra of the $SO(3)$ group closes.

We now turn to the main tasks of removing the phases in such a manner that the resulting final form of V_n is perspicuously exhibited. The precise initial specification of V_n is intimately related to the way in which the phases are treated in our prescription. Two technical points arise, and as only the first is needed in the simplest $n = 2$ case, we again turn to treating the problem iteratively.

In the $n = 2$ case there are phase freedoms

$$D = \begin{pmatrix} d \\ s \end{pmatrix} \rightarrow \exp\left(-\mathrm{i}\xi T_3\right) D \tag{2}$$

and

$$\overline{U} = (\overline{uc}) \rightarrow \overline{U} \exp(\mathrm{i}\omega T_3) \exp\left(\frac{\mathrm{i}}{2}\chi 1\right) \tag{3}$$

where we have reverted to the usual Pauli matrix notation $\tau^i (i = 1, 2, 3)$ for the λ^i, $T_i = \tau_i/2$ and ξ, ω and χ are real. Notice that, as the overall phase is physically irrelevant, there is no χ term in equation (2) corresponding to the one in equation (3). Indeed the overall phase is always trivial to treat, and we have here denoted it by χ to emphasize that it will immediately eliminate the corresponding phase in equation (1). It should now be clear that equation (1) is not the most convenient starting specification for V_2. Clearly a product form

$$V_2 = \exp\left(\frac{-\mathrm{i}}{2}\chi 1\right) \exp(-\mathrm{i}\rho T_3) \exp(-\mathrm{i}\theta_A T_A) \tag{4}$$

where $A = 1, 2$, gives an immediate improvement, exposing the diagonal matrices on the left of the structure. But this can be further exposed by considering the form of the right-hand term, K_2 say. Observe that θ_A can be regarded as the components of a two-dimensional vector rotated by the U_1 factor generated by T_3 in our SU_2. Thus, the form of K_2 can be re-expressed as

$$K_2(\varepsilon, \theta, T) = \exp(-\mathrm{i}\varepsilon T_3) \exp(2\mathrm{i}\theta T_2) \exp(\mathrm{i}\varepsilon T_3) \tag{5}$$

where the parameters ε and θ replace the original θ_A. (The connection between the two sets of parameters is trivial to establish, but is not required here.) Substituting this form back into equation (4) reveals

$$V_2 = \exp\left(\frac{-\mathrm{i}\chi}{2}\right) \exp(-\mathrm{i}[\varepsilon + \rho]T_3) \exp(2\mathrm{i}\theta T_2) \exp(\mathrm{i}\varepsilon T_3) \tag{6}$$

and comparison with equations (2) and (3) shows that the phase changes specified by χ, $\omega = \rho$, and $\xi = \varepsilon$, produce

$$V_2 = \exp(2\mathrm{i}\theta T_2) \tag{7}$$

as our final one parameter description of V_2. The concrete matrix form of V_2 is now

$$V_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8}$$

revealing that θ is the well known Cabibbo [4] angle.

Now we turn to the $n = 3$ case which is currently of most physical interest. To make optimal use of the analysis used so conveniently in the $n = 2$ case above, we propose to exploit the $SU(2) \times U(1)$ subgroup structure of $SU(3)$ so extensively developed some 30 years ago by Lipkin *et al* [10], and the version we present differs from the presentation of Carruthers [11] only by trivial signs specifically introduced for our present interests. With the notation introduced earlier, the SU_2 generators are

$$T_i = \lambda_i/2 \tag{9}$$

exactly as before but extended by a third row and column of zeros, and we introduce

$$T = \frac{1}{2\sqrt{3}}\lambda_8 \tag{10}$$

as the generator of the U_1 which commutes to zero with each of the T_i. It must be emphasized, of course, that the generalization of D is now to a three component column with descending entries d, s and b. Similarly the generalization of $\overline{U}$ is to $(\overline{uc\bar{t}})$. We then introduce another set of SU_2 generators by

$$U_1 = \frac{\lambda_6}{2} \qquad U_2 = \frac{\lambda_7}{2} \qquad U_3 = \frac{\sqrt{3}}{4}\lambda_8 - \frac{1}{4}\lambda_3 \tag{11}$$

and an associated U_1 generator by

$$U = \frac{-1}{4\sqrt{3}}\lambda_8 - \frac{1}{4}\lambda_3 \tag{12}$$

by copying the structure of T_i and T which distinguished the third row and column but now distinguishing the first row and column. Yet again we define generators of a third SU_2 by

$$V_1 = \frac{\lambda_4}{2} \qquad V_2 = \frac{-\lambda_5}{2} \qquad V_3 = -\frac{\sqrt{3}}{4}\lambda_8 - \frac{1}{4}\lambda_3 \tag{13}$$

and an associated U_1 generator by

$$V = \frac{-1}{4\sqrt{3}}\lambda_8 + \frac{1}{4}\lambda_3 \tag{14}$$

by distinguishing this time the second row and column, and judiciously inserting minus signs into V_2 and V_3 for our notational convenience. (We hope that in context there will be no confusion between these components of $\underline{V}$ and the unitary matrix.) It will be noted that the first two components of $\underline{T}$, $\underline{U}$, and $\underline{V}$ give a basis for the six off-diagonal matrices. However, that T_3, U_3, V_3 together with T, U, V and the unit matrix, must really be dependent on only three independent matrices in the diagonal sector. We shall see, nevertheless, that this notion is most suited to our purposes. Therefore we retain this overspecification and record the relationships reflecting the degeneracy for future use. The notation has been designed for maximum symmetry between the three spins, and in particular

$$T_3 + U_3 + V_3 = 0 \tag{15}$$

and

$$T + U + V = 0 \tag{16}$$

with our choice of sign conventions. The remainder of the relationships can be conveniently, but still redundantly, specified in the form

$$T_3 = T + 2V \tag{17}$$
$$U_3 = U + 2T \tag{18}$$
$$V_3 = V + 2U \tag{19}$$

which neatly exposes the cyclical nature of the notation. In practice, as we shall shortly see, the most immediately useful relationships are those expressing the third member of an SU_2 set of generators in terms of two of the singlet operators as in equations (17)–(19) above, or their variants utilizing equation (16) such as

$$V_3 = -V - 2T \tag{20}$$

and

$$2U = -T - T_3 \tag{21}$$

or again their variants using equation (17) such as

$$U_3 = 3U - 2V_3 \tag{22}$$

and

$$2U_3 = 3T - T_3. \tag{23}$$

We are now ready to work through the details of the $n = 3$ case. There are phase freedoms

$$D \to \exp(-i\xi T_3)\exp(-i\gamma T)D \tag{24}$$

$$\overline{U} \to \overline{U}\exp(i\omega T_3)\exp\left(\frac{i\chi}{2}\mathbf{1}\right)\exp(iKT) \tag{25}$$

and we can take

$$V_3 = \exp\left(\frac{-i\chi}{2}\right)\exp(-i\rho T_3)\exp(-i\mu T)K(\varepsilon, \theta, T)K(\delta, \psi, V)K(\lambda, \phi, u) \tag{26}$$

where the matrices K are now (3×3) in an obvious extension of the previous notation. Notice that this extension can conveniently be viewed in the form

$$V_3 = V_2\exp(-i\mu T)K(\delta, \psi, V)K(\lambda, \phi, U) \tag{27}$$

where V_2 has been extended by an extra column and row of zeros, and that the exponential term involving T can then be taken to the left through any part of V_2 since T commutes with all the matrices in V_2. This time things are a little more complicated, and before we can adjust phases it is necessary to consider the structure implied whenever two K matrices are contiguous. In expanded form we see that

$$
\begin{aligned}
K(\varepsilon, \theta, T)&K(\delta, \psi, V) \\
&= \exp(-i\varepsilon T_3)\exp(2i\theta T_2)\exp(i\varepsilon T_3)\exp(-i\delta V_3)\exp(2i\psi V_2)\exp(i\delta V_3)
\end{aligned} \tag{28}
$$

when it becomes evident that there is a 'phase matrix' of $\exp(i\varepsilon T_3)\exp(-i\delta V_3)$ appearing between the two 'rotations' $\exp(2i\theta T_2)$ and $\exp(2i\psi V_2)$. However, it is clear from equation (17) and from equation (20) that this 'phase matrix' can be expressed as $\exp(i[\varepsilon+2\delta]T)\exp(i[\delta+2\varepsilon]V)$, so it can be seen that the left hand factor may be commuted to the left-hand end of V_3, and that the right-hand factor can be commuted one step to the right of the $K(\delta, \psi, V)$ factor in equation (28). The next step is to examine the structure of V_3 farther to the right of the 'rotation matrix' part of $K(\delta, \psi, V)$ in equation (26).

Noting the extra phase we have just moved to the right this part of V now becomes

$$
\begin{aligned}
\exp(i[\delta + 2\varepsilon]V)&\exp(i\delta V_3)K(\lambda, \phi, U) \\
&= \exp(i[\delta + 2\varepsilon]V)\exp(i\delta V_3)\exp(-i\lambda U_3)\exp(2i\phi U_2)\exp(i\lambda U_3).
\end{aligned} \tag{29}
$$

This time it is clear that at least some part of the 'phase matrix' before the ϕ 'rotation matrix' cannot be moved farther to the right. However, we can choose to use equations (19) and (22) to write the first line of equation (29) as

$$\exp\left(2i[\delta + \varepsilon + \lambda]V_3\right)\exp\left(-iU[4\varepsilon + 2\delta + 3\lambda]\right)$$

revealing that it is possible to move the right-hand factor to the right through the ϕ 'rotation matrix' and leaving the residual 'phase matrix' in terms of V_3 alone. The final step is then to express the (now) three phases to the right of equation (26) in terms of T_3 and T. This is easily seen to have the form

$$\exp(iT[\delta + 2\varepsilon + 3\lambda])\exp(iT_3[\delta + 2\varepsilon + \lambda])$$

by using equations (21) and (23).

We can now adjust the phases in equations (22) and (23), so that taking

$$\xi = \delta + 2\varepsilon + \lambda \tag{30}$$

$$\gamma = \delta + 2\varepsilon + 3\lambda \tag{31}$$

$$\omega = \varepsilon + \rho \tag{32}$$

$$K = \mu - \varepsilon - 2\delta \tag{33}$$

and calling

$$\delta + \varepsilon + \lambda = \Delta \tag{34}$$

the form

$$V_3 = \exp(2\mathrm{i}\theta T_2)\exp(2\mathrm{i}\psi V_2)\exp(2\mathrm{i}\Delta V_3)\exp(2\mathrm{i}\phi U_2) \tag{35}$$

emerges as the final four parameter form of the mixing matrix for the $n = 3$ case.

As this is currently thought to be the most important physical case, we pause here to retrieve some well known parametrizations before moving on to higher numbers of generations of quarks.

The first thing to realize is that the particular grouping of rotations and phases which we have presented, although very convenient for counting parameters and demonstrating the principles, is by no means unique. Indeed the very obvious construction

$$V_3 = \exp(2\mathrm{i}\theta_{23}U_2)K(-\delta_{13}, -\theta_{13}, V)\exp(2\mathrm{i}\theta_{12}T_2) \tag{36}$$

where the phase remains on both sides of the 1–3 rotation matrix in K, is precisely the recommended 'standard form' given in reference [5] and credited primarily to Chau and Keung [6]. Expanding into matrix form we see that this is

$$V_3 = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-\mathrm{i}\delta_{13}) \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(\mathrm{i}\delta_{13}) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(\mathrm{i}\delta_{13}) & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(\mathrm{i}\delta_{13}) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(\mathrm{i}\delta_{13}) & c_{23}c_{13} \end{bmatrix} \tag{37}$$

where c_{12} and s_{23} denote respectively $\cos\theta_{12}$ and $\sin\theta_{23}$ etc, as in equation (3) of reference [5]. The axis of rotation is indicated by the missing index.

On the other hand, the original KM matrix [3] is of 'Euler angle' type, involving 'rotations' about only two axes. This time we may write this as

$$V_3 = \exp(-2\mathrm{i}\theta_2 U_2)\exp(-2\mathrm{i}\theta_1 T_2)\exp\left(\frac{\mathrm{i}[\delta + \pi]}{3}\right)\exp(-2\mathrm{i}[\delta + \pi]T)\exp(2\mathrm{i}\theta_3 U_2) \tag{38}$$

where the existence of an overall phase (involving π which has the familiar mathematical value and should not be confused with the four parameters) is needed in our notation to recover equation (4) of reference [5]. This can be expanded as

$$V_3 = \begin{bmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 \exp(\mathrm{i}\delta) & c_1 c_2 s_3 + s_2 c_3 \exp(\mathrm{i}\delta) \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 \exp(\mathrm{i}\delta) & c_1 s_2 s_3 - c_2 c_3 \exp(\mathrm{i}\delta) \end{bmatrix} \tag{39}$$

where this time c_1 denotes $\cos\theta_1$ etc., and the index on the angles shows the axis of rotation directly.

It is now clear from the last example that the complex entries in the CKM matrix can be contained in four positions. This raises the amusing possibility of a description in which the mixing form current to mass eigenstates exactly contrives to put complex entries only in the final row and column thus 'interpreting' the CP violation in the kaon system purely in terms of intermediate top and bottom exchange contributions. One way to achieve this is to take

$$V_3 = \exp(2\mathrm{i}\theta T_2)K(-\tfrac{1}{2}\delta, -\phi, V)\exp(2\mathrm{i}\psi T_2) \tag{40}$$

so that the expanded form

$$V_3 = \begin{bmatrix} \begin{array}{c} [\cos\theta\cos\phi\cos\psi \\ -\sin\theta\sin\psi] \end{array} & \begin{array}{c} [\cos\theta\cos\phi\sin\psi \\ +\sin\theta\cos\psi] \end{array} & \cos\theta\sin\phi\exp(\mathrm{i}\delta) \\ \begin{array}{c} [-\sin\theta\cos\phi\cos\psi \\ -\cos\theta\sin\psi] \end{array} & \begin{array}{c} [\cos\theta\cos\psi \\ -\sin\theta\cos\phi\sin\psi] \end{array} & -\sin\theta\sin\phi\exp(\mathrm{i}\delta) \\ -\cos\psi\sin\phi\exp(-\mathrm{i}\delta) & -\sin\phi\sin\psi\exp(-\mathrm{i}\delta) & \cos\phi \end{bmatrix} \tag{41}$$

shows this feature directly.

Finally, we turn to the description of the CKM matrix in terms of phases of the unitarity triangles [8]. Curiously, the key step [1, 2] in making the connection is to parametrize the CKM matrix so that all the complex terms are in the top left-hand corner. We take the form

$$V_3 = \exp(-2\mathrm{i}\theta T_2)\exp(-2\mathrm{i}\psi V_2)\exp(\tfrac{1}{3}\mathrm{i}\pi[1-6U])\exp(\tfrac{1}{3}\mathrm{i}\varepsilon[1-6V])\exp(2\mathrm{i}\phi T_2) \tag{42}$$

which expanded out reads

$$V_3 = \begin{bmatrix} \begin{array}{c} [-\cos\theta\cos\psi\cos\phi \\ +\sin\theta\sin\phi\exp(\mathrm{i}\delta)] \end{array} & \begin{array}{c} -[\cos\theta\cos\psi\sin\phi \\ +\sin\theta\cos\phi\exp(\mathrm{i}\delta)] \end{array} & \cos\theta\sin\psi \\ \begin{array}{c} [-\sin\theta\cos\psi\cos\phi \\ -\cos\theta\sin\phi\exp(\mathrm{i}\delta)] \end{array} & \begin{array}{c} [-\sin\theta\cos\psi\sin\phi \\ +\cos\theta\cos\phi\exp(\mathrm{i}\delta)] \end{array} & \sin\theta\sin\psi \\ \sin\psi\cos\phi & \sin\psi\sin\phi & \cos\psi \end{bmatrix}. \tag{43}$$

To display the connections to the Kayser [1, 2] form more clearly we define λ by

$$\cos\psi = \lambda\cos\delta \tag{44}$$

and r_{ij}, for $i = u, c$ and $j = d, s$, by

$$r_{ij}\tan\delta = \tan(\arg V_{ij}) \tag{45}$$

where (as we shall soon see directly) the four r_{ij} are related by a single constraint. From the top left hand four entries of V_3 we now see directly that

$$r_{ud} = \frac{\tan\theta\tan\phi}{\tan\theta\tan\phi - \lambda} \tag{46}$$

$$r_{cd} = \frac{\tan\phi}{\tan\phi + \lambda\tan\theta} \tag{47}$$

$$r_{us} = \frac{\tan\theta}{\tan\theta + \lambda\tan\phi} \tag{48}$$

and

$$r_{cs} = \frac{1}{1 - \lambda\tan\theta\tan\phi}. \tag{49}$$

Eliminating λ, ϕ and θ from these equations reveals

$$\frac{(1 - r_{cd})(1 - r_{us})}{r_{cd}r_{us}} = \frac{(1 - r_{ud})(1 - r_{cs})}{r_{ud}r_{cs}} \tag{50}$$

as the constraint equation expected. Then we find

$$\lambda^2 = \frac{(1 - r_{ud})(1 - r_{cs})}{r_{ud}r_{cs}} \tag{51}$$

with alternative expressions yielded by the use of the constraint equation. Reintroducing δ by equation (44) relates $\tan\delta$ to a complicated quotient of sums of products of the tangents of the angles of the unitarity triangles. We do not quote this directly, as we find no simple

expression, although the algebra is direct and straightforward. Finally, we can substitute equations (50) and (51) back into pairs of equations (46) to (49) to reveal

$$\tan^2 \theta = \frac{(1 - r_{cs})r_{us}}{r_{cs}(1 - r_{us})} \tag{52}$$

and

$$\tan^2 \phi = \frac{(1 - r_{us})r_{ud}}{r_{us}(1 - r_{ud})} \tag{53}$$

where alternative expressions are available by using the constraint equation (50) yet again. Now that λ and δ are known (at least implicitly), equation (44) gives ψ to complete the connection between our parametrization and the unitary triangle angles of Kayser [1, 2]. We find the algebraic complexity disappointing, but the connections are at least clearly made.

We finally treat all cases with four or more generations. Consider expanding from n to $n + 1$ where $n \geqslant 3$. The first two new matrices introduced will be λ matrices whose indices are n^2 and $n^2 + 1$, and which have entries 1 and $-i$ respectively in the top right-hand corner, with their conjugates appearing in the bottom left-hand corner. To enable easy visualization we denote these as $2\Sigma_1$ and $2\Sigma_2$, where the n-dependence has been suppressed. Clearly Σ_1 and Σ_2 are a part of an $SU(2)$ set of generators, the third member of which we call Σ_3 with entries $\frac{1}{2}$ and $-\frac{1}{2}$ in the top left-hand corner and the bottom right-hand corner respectively. Again, consider the new diagonal matrix introduced by the expansion from n to $n + 1$. It is, of course, $\lambda_{n(n+2)}$. This has entries $[2/n(n + 1)]^{1/2}$ down the first n diagonal places, and $(-)[2n/n + 1]^{1/2}$ in the final diagonal place.

Obviously $\lambda_{n(n+2)}$ commutes with the whole set of $U(n)$ matrices parameterizing V_n, and we now write (in an obvious extension of equation (27))

$$V_{n+1} = V_n \exp\left(\frac{-i\nu\lambda_{n(n+2)}}{2\sqrt{3}}\right) K(\eta, \zeta, \Sigma) \ldots \tag{54}$$

where there are now n new K factors implied, of which we have shown only the first explicitly. As previously, the exponential factor can be moved to the left as required. Now the first new K factor can be expanded as before in the form

$$K(\eta, \zeta, \Sigma) = \exp(-i\eta\Sigma_3) \exp(2i\zeta\Sigma_2) \exp(i\eta\Sigma_3) \tag{55}$$

and the now familiar task is to remove the first exponential factor by expressing it in terms of diagonal matrices which either commute with everything to the left or through at least one term to the right. Our method is a straightforward extension of that used in the $n = 3$ case. We introduce a diagonal matrix C_{n+1} which has entries $\frac{1}{2}[(n - 1)/3(n + 1)]^{1/2}$ in the top left-hand corner and the bottom right-hand corner, and entries $(-)[1/3(n - 1)(n + 1)]^{1/2}$ in the remaining $(n - 1)$ diagonal places. This has been designed to be traceless, and to be normalized so that the trace of its square is the same as the corresponding matrices in the 3×3 case. It is trivial to see that

$$\Sigma_3 = \left[\frac{n}{2(n + 1)}\right]^{1/2} \lambda_{n(n+2)} + \left[\frac{3(n - 1)}{(n + 1)}\right]^{1/2} C_{n+1}. \tag{56}$$

Obviously $\lambda_{n(n+2)}$ commutes with the entire structure to the left of $K(\eta, \zeta, \Sigma)$ in equation (54), and also C commutes with the $\underline{\Sigma}$ which appear to its right in equation (55). This latter point is, of course, by construction in analogy with the 3×3 case, but is perhaps intuitively even easier to see now that the matrices are more sparse.

A last word should probably be said concerning the counting of parameters in the general case as displayed by the present analysis. As we construct V_{n+1}, working from the left in equation (54) we first encounter V_n with $(n - 1)^2$ independent real parameters

conveniently viewed as $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}(n-1)(n-2)$ phases. Next we find n factors of K, each having the structure shown in equation (55), namely that of a rotation surrounded by exponential phase factors. Finally, there is a phase factor carried on the new diagonal matrix introduced at this level. What we have shown however is that the phase to the left of the first new K factor may be removed by expressing it in terms of a part which commutes to the left to be absorbed on phases of $\overline{U}$, and part which commutes one step to the right. Finally the phase to the extreme right of the new K factors may be absorbed into the phases of D. Thus, overall, there are n new rotation parameters, and $(n-1)$ new phases. The number of angles then becomes $\frac{1}{2}n(n-1) + n = \frac{1}{2}n(n+1)$, and the number of phases $\frac{1}{2}(n-1)(n-2) + (n-1) = \frac{1}{2}n(n-1)$ as previously stated. Perhaps it should be emphasized that, just as in the $n=3$ case, there are many possible variants of representation of V_n and arising in much the same way. We do not expand on this theme, however, since currently the physical interest is in the $n=3$ case and there is no evidence of further generations of quarks and leptons. One of the authors (KJB) still retains a hope that a further generation will be found and that the economy of orthogonal organizing symmetries [12] will be utilized by nature. In that event, the analysis presented here would be immediately utility in describing possible mass breaking schemes.

Acknowledgments

References

[1] Kayser B 1994 *'94 Electroweak Interactions and Unified Theories* ed J Tran Thanh Van (Gif-sur-Yvette: Edition Frontieres) p 523
[2] Aleksan R, Kayser B and London D 1994 *Phys. Rev. Lett.* **73** 18
[3] Kobayashi M and Maskawa T 1973 *Prog. Theor. Phys.* **49** 652
[4] Cabibbo N 1963 *Phys. Rev. Lett.* **10** 531
[5] Review of particle properties 1994 *Phys. Rev.* D **50** 1173 (See particularly references [6] and [7] given below, and references [4–7] of this review)
[6] Chau L-L and Keung N-Y 1984 *Phys. Rev. Lett.* **53**, 1802
[7] Botella F J and Chau L-L 1986 *Phys. Lett.* **168B** 97
[8] See references [38] of reference [5] above
[9] Gell-Mann M 1962 *Phys. Rev.* **125** 1067
[10] Levinson C A, Lipkin H J and Meshkov S 1962 *Phys. Rev. Lett.* **1** 44
[11] Carruthers P 1966 *Introduction to Unitarity Symmetry* (New York: Interscience)
[12] Barnes K J, Dondi P H, Jarvis P D and Ketley I J 1979 *Phys. Lett.* **80B** 302; 1979 *J. Phys. G: Nucl. Phys.* **5** 1